

# Dynamics for Density Operator Interpretations of Quantum Theory

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## Abstract

We first introduce and discuss density operator interpretations of quantum theory as a special case of a more general class of interpretations, giving special attention to a version that we call the ‘atomic version’. We then review some crucial parts of the theory of stochastic processes (the proper context in which to discuss dynamics), and develop a general framework for specifying a dynamics for density operator interpretations. This framework admits infinitely many empirically equivalent dynamics. We give some examples, and discuss some of the properties of one of them.

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## 1 Density Operator Interpretations as Modal Interpretations

### 1.1 Introduction to Modal Interpretations

One way to ‘interpret’ quantum mechanics is to say, for any system, what properties are possessed, or what observables have a definite value. The theorem of Kochen and Specker (1967) shows that, under certain constraints (which we will take for granted), one cannot take *every* observable to have a definite value. Hence the central interpretive question: Which observables have a definite value?

There are in the literature a number of proposed answers to this question,<sup>1</sup> the aim of which is to prescribe, at each time, a set of possible properties (or, equivalently, a set of definite-valued observables), *i.e.*, those properties that may be possessed by a system (as opposed to the properties, familiar from standard quantum mechanics, that are neither possessed nor not possessed). In addition, of course, one needs a probability measure over the possible properties. We are thus given an answer to any question of the form: At time  $t$ , what properties may be possessed, and for any such property, what is the probability that it is possessed?

These ‘properties that may be possessed’, the possible properties of a system at a time, are probably best required to form either an algebra or a partial algebra under the quantum-algebraic (lattice-theoretic) operations of meet, join, and orthocomplement. After all, probability measures are generally taken to be defined over *algebras* (or partial algebras) of events.<sup>2</sup> As it happens, many proposals choose an algebra of a certain kind, dubbed a faux-Boolean algebra. These algebras are constructed (in Hilbert space) as follows. Choose any set,  $S$ , of mutually orthogonal subspaces in the Hilbert space. Let  $S^\perp$  be the set of all subspaces orthogonal to the span of  $S$ . The algebraic closure (under meet, join, and orthocomplement) of  $S \cup S^\perp$  is a faux-Boolean algebra. We will mainly be concerned with proposals that choose for their algebra of possible properties a faux-Boolean algebra, though we will point out when our results generalize to other proposals.

One must, of course, adopt the quantum-mechanical probability measure over the algebra of possible properties. That is, for any system with the state (statistical operator),  $\hat{W}$ , the probability that the system possesses the property  $P$  is given by

$\text{Tr}[WP]$ , where  $P$  is, of course, an element in the system's faux-Boolean algebra of possible properties. (We do not distinguish notationally among projection operators, subspaces, and properties.) Moreover, all extant proposals of the sort considered here (*i.e.* that choose a faux-Boolean algebra) choose  $S$  so that every property in  $S^\perp$  has probability zero. That is, all of these proposals make the faux-Boolean algebra of properties state-dependent, and do so in such a way that every element of  $S^\perp$  is in the null space of  $W$ . (This requirement guarantees that the probability theory generated by modal interpretations is effectively classical. See Dickson (1995a, 1995b) for discussion. See also Bell and Clifton (1995) and Zimba and Clifton (1998) for discussions of the algebraic structure of possible properties.)

For every property,  $P$ , in a faux-Boolean algebra,  $\mathcal{B}$ , of possible properties, then, we may say that  $P$  is either possessed or not by a system (at a given time). The complete set of possessed properties for a given system (at a given time) is therefore a map,  $m : \mathcal{B} \rightarrow \{0, 1\}$  ('0' for 'not possessed' and '1' for 'possessed'). It is easy to show that, because we have adopted the quantum-mechanical probability measure over  $\mathcal{B}$ ,  $m$  must be an ultrafilter on  $\mathcal{B}$ . That is,  $m$  must map one and only one atom,  $P_{\text{atom}}$ , of  $\mathcal{B}$  to 1, and for every other element,  $P$ , of  $\mathcal{B}$ ,  $m(P) = 1$  if and only if  $PP_{\text{atom}} = P_{\text{atom}}$  (or, in terms of subspaces,  $P_{\text{atom}} \subseteq P$ ). One can therefore specify the *complete* state of a system by specifying an atom in its faux-Boolean algebra. Moreover, as we mentioned, with probability 1 the complete state of a system will be specified by an element of the set,  $S$ , used in the construction of its faux-Boolean algebra. Henceforth, therefore, we will mostly restrict our attention to the elements of  $S$ , and we refer to the actually possessed element of  $S$  as 'the complete state' of a system.

## 1.2 The Atomic Version

We will concentrate our attention on a proposal that we call the 'atomic version', but we begin with the version due to Dieks (1988, 1989) and its generalization by Vermaas and Dieks (1995). (These are related to versions due to Kochen (1985) and Healey (1989). See note 1 for further references.) In the above terminology, Vermaas and Dieks construct a set,  $S(t)$ , at each time,  $t$ , as follows: for any system with a quantum-mechanical state,  $W(t)$ , the elements of  $S(t)$  are the elements that correspond to non-zero eigenvalues in the (unique) spectral resolution of  $W(t)$ .

The apparent simplicity of this proposal is deceiving. The Vermaas–Dieks proposal is meant to apply to *every* system individually. We should therefore ask: What *counts* as a 'system'? Possibly, Dieks' original intuition was that every Hilbert space,  $\mathcal{H}^\nu$  that appears in *some* factorization,  $\mathcal{H}_{\text{univ}} = \mathcal{H}^\alpha \otimes \mathcal{H}^\beta \otimes \dots \mathcal{H}^\omega$  of the Hilbert space,  $\mathcal{H}_{\text{univ}}$ , of the universe corresponds to a system, in the above sense. (The index  $\nu$  ranges over the Greek superscripts  $\alpha, \beta, \dots$ ) One then uses the statistical operator for each such subsystem (found by tracing out the rest of the universe), to establish a faux-Boolean algebra for that subsystem. However, it has been shown (Bacciagaluppi 1995) that this option leads to a Kochen–Specker contradiction. Apparently, the only way around this contradiction is to adopt some form of contextuality. (See Bacciagaluppi and Vermaas (1997) for further discussion.) Dieks (1997) has, in fact, chosen to follow Healey in supposing that there is a *preferred factorization* of the Hilbert space of the universe (though for Dieks, the preferred factorization should be chosen, in a sense pragmatically, to depend on the relevant interactions). Subsystems of the universe then correspond to the factor spaces that appear in the preferred factorization of  $\mathcal{H}_{\text{univ}}$ .

As an aside, we note that the idea of a preferred factorization is not, perhaps, as *ad hoc* as it might at first appear. After all, assuming that the universe is really made up of, say, electrons, quarks, and so on, it makes good sense to take these objects to be the ‘real’ constituents of the universe, *i.e.*, the bearers of properties that do not necessarily supervene on the properties of subsystems. Indeed, it would appear strange, if not downright silly, to suppose that, for example, a ‘system’ composed of the spatial degrees of freedom of some electron and the spin degrees of freedom of some atom is a genuine subsystem of the universe, deserving of its own properties (apart from those properties that it inherits by virtue of its being composed of two other systems).

However, even this proposal runs into difficulties. Consider, for example, a system whose Hilbert space is  $\mathcal{H}^{\alpha\beta} = \mathcal{H}^\alpha \otimes \mathcal{H}^\beta$ , and one of its subsystems, whose Hilbert space is  $\mathcal{H}^\alpha$ . (We refer to such systems and subsystems by the superscripts that label the Hilbert spaces,  $\alpha, \beta, \alpha\beta$ .) It can easily happen that a spectral projection,  $P^{\alpha\beta}$ , of  $W^{\alpha\beta}$  (the quantum-mechanical state of  $\alpha\beta$ ) does not commute with  $P^\alpha \otimes \text{I}\text{l}^\beta$ , where  $P^\alpha$  is a spectral projection of  $W^\alpha$  and  $\text{I}\text{l}^\beta$  is the identity on  $\mathcal{H}^\beta$ . But then it is not obvious that we can escape a Kochen–Specker contradiction (and, indeed, Clifton (1996) has derived one), or even that we can define a joint probability for  $\alpha\beta$  to possess  $P^{\alpha\beta}$  and  $\alpha$  to possess  $P^\alpha$ . In fact, it is well known that there is, in general, no expression for the joint probability for non-commuting projections that is valid in every quantum-mechanical state. (Whether or not we wish to say that  $\alpha\beta$  possesses  $P^\alpha \otimes \text{I}\text{l}^\beta$  whenever  $\alpha$  possesses  $P^\alpha$ —an assumption necessary for deriving a Kochen–Specker contradiction—a joint probability for  $\alpha\beta$  to possess  $P^{\alpha\beta}$  and  $\alpha$  to possess  $P^\alpha$  will induce a joint probability measure for  $P^{\alpha\beta}$  and  $P^\alpha \otimes \text{I}\text{l}^\beta$ .)

On the other hand, we do not need a general expression for the joint probability of any two non-commuting projections, but only expressions that are valid for limited sets of non-commuting projections. Nevertheless, while some have tried to find such expressions, and have succeeded in special cases (Vermaas 1996), no generally acceptable expression has yet been found. Indeed, recent results (Vermaas 1997) suggest that no satisfactory expression will be found.

Because of these problems, we propose to adopt a still more conservative approach, which we call the ‘atomic version’. This version adopts a preferred factorization of the Hilbert space for the universe, and assigns faux-Boolean algebras of possible properties directly only to those subsystems corresponding to atomic factors in the preferred factorization (*i.e.*, those that are not themselves tensor products of factors appearing in the preferred factorization). All other subsystems inherit properties from these ‘atomic’ subsystems by the principle of property composition: if two subsystems,  $\alpha$  and  $\beta$ , possess the properties  $P_m^\alpha$  and  $P_n^\beta$ , respectively, then the system composed of  $\alpha$  and  $\beta$  possesses the property  $P_m^\alpha \otimes P_n^\beta$ .

For example, suppose that the Hilbert space for the universe has the preferred factorization  $\mathcal{H}_{\text{univ}} = \mathcal{H}^\alpha \otimes \mathcal{H}^\beta \otimes \dots \otimes \mathcal{H}^\omega$ . The atomic version assigns a faux-Boolean algebra to each of the subsystems  $\alpha, \dots, \omega$ . Following Vermaas and Dieks (1995), it does so by letting  $S^\nu$  (the set used to generate the faux-Boolean algebra,  $\mathcal{B}^\nu$ , for the atomic system,  $\nu$ ) be the set of projections with non-zero eigenvalues in the (unique) spectral resolution of  $W^\nu$ , where  $W^\nu$  is the quantum-mechanical state of the atomic system,  $\nu$ .

As we said, in the atomic version, all compound systems have only those properties that they inherit from their constituent atomic subsystems. Therefore, the algebra of properties for a compound system is just the Cartesian product of the algebras of properties for the constituent atomic subsystems. To complete the atomic version, therefore, we need only the joint probabilities for properties of all subsystems. The other joint probabilities are obtained by additivity. As Vermaas and Dieks (1995) have

already noted, the obvious candidate is

$$\Pr(P_{i_\alpha}^\alpha, P_{i_\beta}^\beta, \dots, P_{i_\omega}^\omega) = \text{Tr}[WP_{i_\alpha}^\alpha \otimes P_{i_\beta}^\beta \otimes \dots \otimes P_{i_\omega}^\omega]. \quad (1)$$

From this expression, it is clear that while atomic systems  $\alpha$  and  $\beta$  might have non-zero probability to possess the properties  $P^\alpha$  and  $P^\beta$ , respectively, it might still happen that the compound system  $\alpha\beta$  has zero probability to possess the property  $P^\alpha \otimes P^\beta$ . (Think, for example, of the singlet state of two spin-1/2 particles.) In this case one might like to say that  $P^\alpha \otimes P^\beta$  should *not* be in the set,  $S^{\alpha\beta}$ , used to generate the faux-Boolean algebra of properties for the compound system  $\alpha\beta$ . However, such a policy would make the problem of finding a dynamics even harder than it already is. The reason is that finding a dynamics is made more difficult when the cardinality of the set,  $S^{\alpha\beta}$ , can change in time. (This point should become clear in Sections 2 and 3.) Hence we adopt the policy that the faux-Boolean algebra of properties for a compound system is indeed generated by the set

$$S := \left\{ P = P_{i_\alpha}^\alpha \otimes \dots \otimes P_{i_\omega}^\omega \mid P_{i_\alpha}^\alpha \in S^\alpha, \dots, P_{i_\omega}^\omega \in S^\omega \right\}, \quad (2)$$

where the Hilbert space for the compound system is  $\mathcal{H} = \mathcal{H}^\alpha \otimes \dots \otimes \mathcal{H}^\omega$ ,  $\alpha, \dots, \omega$  are atomic systems, and for all  $\nu$ ,  $S^\nu$  is the set used to generate the faux-Boolean algebra of properties for the atomic system  $\nu$ . This construction differs from the perhaps more natural one, given by the additional constraint that  $\text{Tr}[WP] \neq 0$  (where  $W$  is the state of the compound system), only on properties that have zero probability.

To finish this discussion, and to foreshadow the discussion of Section 3, note that in general the sets  $S^\nu$  are time-dependent. Our work here may therefore be seen as a generalization of work by Bohm (1952), Bell (1987), Vink (1993), and Bub (1996, 1997). They consider a dynamics for a ‘preferred observable’,  $R$ , considered to have a definite value at all times. For us, the preferred observable is time-dependent. That is, we may take it to be some observable,  $R(t)$ , whose eigenprojections are the elements of  $S(t)$  (the set  $S$  as given by (2) at some time,  $t$ ), and whose eigenvalues are arbitrary. Indeed, as we shall see, our work applies to any time-dependent preferred observable whose eigenprojections and eigenvalues evolve differentiably. (We return to this point in greater detail later.)

There is plenty more one might say about the atomic version, but we will not linger over its motivation or consequences here. As we said above, our results will be geared mainly towards the atomic version, though we will note how and under what conditions our results can be applied to non-atomic versions.

(Lest we appear overly optimistic about the prospects for the atomic version, we note that one outstanding problem has not been solved: we have not shown that it will attribute ‘the right’ properties to macroscopic objects. For example, we have not shown that it entails that macroscopic objects are well-localized. This question has been addressed in detail, with ultimately negative results, for non-atomic interpretations.<sup>3)</sup>)

### 1.3 The Problem

Thus far we have specified, for every time  $t$ , the set of possible modal states,  $S(t)$ , and a probability measure over this set. However, one wants to know more. We take

it to be crucial to answer dynamical questions, of the form: Given that a system possesses property  $P$  at time  $s$ , what is the probability that it will possess property  $P'$  at time  $t$  ( $t > s$ )? In other words, we need a *dynamics* of possessed properties.

Some may consider a dynamics of possessed properties to be superfluous. After all, could quantum mechanics not get away with just single-time probabilities? Why can we not settle for an interpretation that supplements standard quantum mechanics *only* by providing in a systematic way a set (the set of possible properties) over which single-time probabilities are defined? If we require of this set that it include the everyday properties of macroscopic objects,<sup>4</sup> then what more do we need?

What we need is an assurance that the *trajectories* of possessed properties are, at least for macroscopic objects, more or less as we see them to be. For example, we should require not only that the book at rest on the desk has a definite location, but also that, if undisturbed, its location relative to the desk does not change in time. Hence one cannot get away with simply specifying the definite properties at each time. We need also to be shown that this specification is at least *compatible* with a reasonable dynamics. Even better, we would like to see the dynamics explicitly.<sup>5</sup> (As we will note below, it is trivial to define an ‘unreasonable’ dynamics, namely, one in which there is no correlation from one time to the next. In such a case, the book on the table might *not* remain at rest relative to the table, even if undisturbed. We take it that such dynamics are not very interesting, and fail to provide any assurance that we can describe the world more or less as we think it is.<sup>6</sup>)

Our main task in this paper is to show how to construct a dynamics describing time-evolution of the complete state of a system. We will investigate some features of one example, with an eye towards suggesting that there are likely to be at least some dynamics that are ‘reasonable’.

The structure of the rest of this paper is as follows. In the next section, we will discuss the general framework of stochastic processes, the appropriate mathematical theory for the rigorous description of a dynamics. In particular, we will discuss the question of how to find finite-time transition probabilities. This problem is much-discussed among theorists of stochastic processes. In Section 3, we develop a general framework for the description of dynamics for complete states. Here we show how to specify infinitesimal parameters that are guaranteed to return the single-time probabilities already defined via quantum theory. In Section 4, we discuss some constraints on dynamics first derived by Vermaas (1996), and we give some examples of dynamics, spending particular attention on one that we call the generalized Schrödinger dynamics. In Section 5 we discuss some of the properties of the generalized Schrödinger dynamics, and we mention some problems for future research. (A review of research on dynamics will appear in Bacciagaluppi (1998a).)

## 2 Preparatory Discussion of Dynamics

### 2.1 Stochastic Processes

It is evident that the dynamics we are after will be genuinely probabilistic. This point can be seen in a trivial example. Let

$$|\Psi(t)\rangle := \sum_i c_i(t)|\alpha_i\rangle \otimes |\beta_i\rangle, \quad (3)$$

where  $|\alpha_i\rangle \in \mathcal{H}^\alpha$  and  $|\beta_i\rangle \in \mathcal{H}^\beta$ . Then the spectral resolution of  $W^\alpha(t)$  is at all times given by  $\{P_i^\alpha\}$  ( $= \{|\alpha_i\rangle\langle\alpha_i|\}$ ), unless there happens to be a degeneracy (in which case the projections in the unique spectral resolution of  $W^\alpha$  would be given by sums of elements of  $\{P_i^\alpha\}$ )—we will assume for this example that there are no degeneracies. But the probabilities attached to these spectral projections will change, due to the time-dependence of the  $c_i(t)$ . Now, consider an ensemble of systems, all with state vector  $|\Psi(t)\rangle$  and with modal states distributed across  $\{P_i^\alpha\}$  according to the quantum-mechanical probability measure,  $|c_i(t)|^2$ . As the distribution,  $|c_i(t)|^2$ , changes in time, some members of the ensemble must make transitions among the  $\{P_i^\alpha\}$  in order to preserve the distribution.

Why can these transitions not be deterministic? Because, assuming that the complete state specifies completely the physical state of the system, there can be nothing to distinguish those systems that make a transition from, say,  $P_1^\alpha$  to, say,  $P_2^\alpha$  from those systems that make a transition from  $P_1^\alpha$  to some property other than  $P_2^\alpha$  (or make no transition at all). There is therefore nothing in the theory that could have ‘determined’ the transition from  $P_1^\alpha$  to  $P_2^\alpha$ .

Deterministic ‘equations of motion’ for the complete states are therefore out of the question. Instead, we need stochastic equations of motion. The most general and powerful framework within which to find and study such equations is the theory of stochastic processes. We shall therefore put our question into this framework, allowing us to borrow and adapt some well-known results from that theory.

We remind the reader of some basic definitions. (There are many books on the topic. Classic texts are Doob (1953) and, in particular, Feller (1968).) A **random variable**,  $V$ , is a map from a sample space (the set of all atomic events in some algebra of events) to the real numbers. The probability measure over (the algebra generated by) the sample space induces a probability measure over (the algebra generated by) the range of  $V$ . We therefore ignore the sample space itself, and consider directly the probability measure over the range of  $V$ , also called the **state space** of  $V$ , writing:

$$p_i^V := p(V = i), \quad (4)$$

defined for all  $i$  in the state space of  $V$ . Because the context makes the meaning clear, we will usually omit the superscripted  $V$ , writing  $p_i$ .

A **stochastic process**,  $V_t$  is an indexed family of random variables with a common state space. For us, the index  $t$  will be continuous, and it will represent time. Moreover, for us, the state space will be a set of integers,  $I$ . Single-time probability distributions are written:

$$p_i^{V_t}(t) := p(V_t = i) \quad (5)$$

for all  $i$  in the state space,  $I$ , of  $V_t$ . Again, we usually write  $p_i(t)$ .

Given a complete set of finite-time transition probabilities, that is, the joint distribution functions,

$$p(V_{t_1} = i_1, V_{t_2} = i_2, \dots, V_{t_n} = i_n), \quad (6)$$

for every set of times,  $\{t_1, \dots, t_n\}$  (where of course  $i_m \in I$  for all  $m$  in  $\{1, \dots, n\}$ ), we can define the finite-time transition probabilities in the usual way by

$$p(V_t = i | V_{t_1} = i_1, \dots, V_{t_n} = i_n) := \frac{p(V_{t_1} = i_1, \dots, V_{t_n} = i_n, V_t = i)}{p(V_{t_1} = i_1, \dots, V_{t_n} = i_n)}, \quad (7)$$

where  $t > t_m$  for all  $m$  in  $\{1, \dots, n\}$ . Equation (7) represents the most general form for the finite-time transition probabilities, and without making any further restrictions, a complete dynamics requires a complete set of such probabilities. However, we will restrict attention here to Markov processes, that is, processes that obey the Markov property:

**Markov Property:** Whenever  $t_1 < t_2 < \dots < t_n < t$ ,

$$p(V_t = i | V_{t_1} = i_1, \dots, V_{t_n} = i_n) = p(V_t = i | V_{t_n} = i_n). \quad (8)$$

We therefore need to find only a complete set of **transition probability functions**,

$$p_{ji}(t, s) := p(V_t = j | V_s = i), \quad (9)$$

for all  $s \leq t$ . We require, for obvious reasons, that  $p_{ii}(t, t) = 1$  and  $p_{ji}(t, t) = 0$  for  $i \neq j$ . (Note that the transition probability function is read from right to left:  $p_{ji}(t, s)$  is the probability of a transition *from*  $i$  at time  $s$  *to*  $j$  at time  $t$ .)

Now, one could try to specify directly all of the  $p_{ji}(t, s)$ , but this strategy is too unwieldy. It is more common, and more convenient in the end, to begin by defining the so-called infinitesimal parameters (defined in the next subsection); then one uses the infinitesimal parameters to build up the finite-time transition probabilities.

Of course, there is one trivial way to specify directly all of the  $p_{ji}(t, s)$ . It is the case of no correlation over time. That is, for all  $s, t, i, j$ , define

$$p_{ji}(t, s) := p_j(t). \quad (10)$$

Such transition probabilities describe the ‘unreasonable’ dynamics that we mentioned above. They obviously return the correct single-time probabilities.

## 2.2 The Infinitesimal Parameters

As we said, we are seeking a dynamics that is more reasonable than (10), and the best way to do so is to begin with the infinitesimal parameters, defined below. However, this procedure brings with it some complications that require further discussion. In this section, we first give some definitions, and then discuss the problems that one encounters when trying to construct finite-time transition probabilities from the infinitesimal parameters. This problem is a standard one in the theory of Markov processes. Our discussion will therefore be brief, and proofs will be omitted. (One can find discussions of the results that we quote in standard books on stochastic processes. See, for example, Doob (1953), Feller (1968), and Gikhman and Skorokhod (1974–1979).)

The **infinitesimal parameters**, of a stochastic process (not necessarily Markovian) are defined, for  $j \neq i$ , as the quantities

$$t_{ji}(t) := \lim_{\varepsilon \rightarrow 0} \frac{p_{ji}(t + \varepsilon, t) - p_{ji}(t, t)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{p_{ji}(t + \varepsilon, t)}{\varepsilon}, \quad (11)$$

if the limit on the right-hand side exists in  $[0, \infty]$ . (We have used  $p_{ji}(t, t) = 0$  for  $j \neq i$ .) For  $j = i$  they are

$$t_{ii}(t) := \lim_{\varepsilon \rightarrow 0} \frac{p_{ii}(t + \varepsilon, t) - p_{ii}(t, t)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{p_{ii}(t + \varepsilon, t) - 1}{\varepsilon}, \quad (12)$$

when the limit exists in  $[-\infty, 0]$ . (We have used  $p_{ii}(t, t) = 1$ .)

From this definition of  $t_{ji}$ , it follows that

$$\sum_j t_{ji}(t) = 0. \quad (13)$$

Intuitively,  $t_{ji}(t)$  (for  $j \neq i$ ), which is always positive, is the rate at which  $j$  gains probability at the expense of  $i$ , while  $t_{ii}(t)$  (which is always negative) is the rate at which  $i$  loses probability. Equation (13) expresses the conservation of probability.

There is an alternative motivation of the infinitesimal parameters. Define  $t_i(t)dt$  to be the probability that the process makes a transition from state  $i$  during the interval  $[t, t + dt]$ , assuming it is in the state  $i$  at time  $t$ . Define  $\Pi_{ji}(t)$  to be the probability of a jump from  $i$  to  $j$  during the same time interval, given that the process is in the state  $i$  at time  $t$  and that a transition takes place. It follows directly from the definitions above (when the  $t_{ji}$  are finite) that

$$t_i(t) = -t_{ii}(t) \quad (14)$$

and

$$\Pi_{ji}(t) = \frac{t_{ji}(t)}{t_i(t)}. \quad (15)$$

(Note that  $\sum_j \Pi_{ji}(t) = 1$ .)

Equation (14) establishes a connection between the infinitesimal parameters and the waiting time in state  $i$ . In particular, one can show that the waiting time in state  $i$  after having arrived there at time  $s$  is exponentially distributed with parameter  $t_i(u)$ . In other words, the probability that a system that arrived in the state  $i$  at time  $s$  will remain there at least until time  $t$  is

$$p_{ii}(t, s) = \exp \left[ - \int_s^t t_i(u) du \right]. \quad (16)$$

Using (14)–(16), one can already see a connection between the infinitesimal parameters and the finite-time transition probabilities, by using  $t_i(t)$  and  $\Pi_{ji}(t)$  to reconstruct the sample paths (realizations) of the process. We imagine that the process begins in the state  $i$ . It waits in that state for some amount of time given (probabilistically) by (16). When it makes a transition at time  $t$ , the probability that it will go to the state  $j$  is  $\Pi_{ji}(t)$ . Once in state  $j$ , it waits for some amount of time, given (probabilistically) by (16), and so on. In this way, one can reconstruct all of the sample paths of the process, and using some results from measure theory, one can

in fact (under certain conditions) reconstruct the finite-time transition probabilities from the knowledge of these sample paths.

However, we will follow a different route, due mainly to Kolmogorov (1931) and Feller (1940). To begin, note that the transition probabilities of a Markov process obey the so-called Chapman–Kolmogorov equations:

$$p_{ji}(t, s) = \sum_k p_{jk}(t, s + h)p_{ki}(s + h, s) \quad (17)$$

for all  $s \leq s + h \leq t$ . The intuitive idea is just that we can ‘sum over’ an intermediate state of the process. From the Chapman–Kolmogorov equations, one can obtain as ‘limit’ equations the so-called forward and backward Kolmogorov equations. We state the theorem here without proof.

**Theorem 1** (Kolmogorov 1931): If the infinitesimal parameters,  $t_{ji}(t)$ , for a Markov process,  $V_t$ , are well-defined, finite, and continuous for all  $t$ , then the transition probability functions,  $p_{ji}(t, s)$ , for  $V_t$  are partially differentiable in  $t$  and  $s$ , and the following hold:

$$\frac{\partial}{\partial t} p_{ji}(t, s) = \sum_k t_{jk}(t)p_{ki}(t, s), \quad (18)$$

$$\frac{\partial}{\partial s} p_{ji}(t, s) = - \sum_k p_{jk}(t, s)t_{ki}(s). \quad (19)$$

These two equations are called the **forward** and **backward Kolmogorov equations**, respectively.

Notice that, because by Theorem 1 the  $p_{ji}(t, s)$  are partially differentiable and  $p_{ji}(t, t) = \delta_{ji}$ , one has

$$t_{ji}(t) = \frac{\partial}{\partial t} p_{ji}(t, s) \Big|_{s=t}. \quad (20)$$

Equation (20) justifies our earlier intuitive interpretation of (11) and (12).

Note that Theorem 1 does not yet guarantee that, given a set of (well-defined, finite, and continuous) infinitesimal parameters,  $t_{ji}(t)$ , we can find a Markov process and a set of finite-time transition probabilities for that process consistent with the  $t_{ji}(t)$ . Theorem 1 already presumes the existence of a Markov process with finite-time transition probabilities  $p_{ji}(t, s)$ , and states two relations (the backwards and forwards equations) between these probabilities and the infinitesimal parameters. We must therefore now ask about the existence of solutions of the Kolmogorov equations given a set of infinitesimal parameters. Feller answered this question by showing how (under certain conditions) to construct a so-called ‘minimal’ solution to the Kolmogorov equations that is consistent with the given infinitesimal parameters. We will review Feller’s construction below, but first we state the theorem.

**Theorem 2** (Feller 1940): 1. Let  $t_{ji}(t)$  be a set of continuous, finite but possibly unbounded functions on a finite or infinite open interval  $T_1 < t < T_2$ , satisfying

$$t_{ji} \geq 0, \quad j \neq i, \quad (21)$$

and

$$\sum_j t_{ji}(t) = 0. \quad (22)$$

Then there exist absolutely continuous functions  $p_{ji}(t, s)$  on  $T_1 < s < t < T_2$  such that

$$0 \leq p_{ji}(t, s) \leq 1, \quad (23)$$

$$\sum_j p_{ji}(t, s) \leq 1, \quad (24)$$

and such that the two Kolmogorov equations are satisfied, as well as the Chapman–Kolmogorov equations. Further, one has

$$\lim_{\varepsilon \rightarrow 0^+} p_{ji}(t, t - \varepsilon) = \delta_{ji}. \quad (25)$$

2. Under certain conditions, in particular if the index set  $I$  is finite, one has

$$\sum_j p_{ji}(t, s) = 1. \quad (26)$$

For any other solution  $q_{ji}(t, s)$  of the Kolmogorov equations satisfying  $\lim_{\varepsilon \rightarrow 0^+} q_{ji}(t, t - \varepsilon) = \delta_{ji}$  one has

$$q_{ji}(t, s) \geq p_{ji}(t, s). \quad (27)$$

Thus, in this case, the ‘minimal’ solution  $p_{ji}(t, s)$  is the unique solution of the Kolmogorov equations with  $\lim_{\varepsilon \rightarrow 0^+} p_{ji}(t, t - \varepsilon) = \delta_{ji}$  and  $\sum_j p_{ji}(t, s) \leq 1$ , and hence the only solution that can be interpreted as a set of transition functions.

We review Feller’s construction of the minimal solution here because it is fairly intuitive. Feller defines

$$p_{ji}^{(0)}(t, s) := \delta_{ji} \exp \left[ - \int_s^t t_i(u) du \right], \quad (28)$$

which can be interpreted as the probability that a system in the state  $i$  at time  $s$  will be in the state  $j$  at time  $t$ , having made no transitions during the interval  $[s, t]$ . (Recall the discussion of (16).) By a careful enumeration of the possibilities, one can extend (28) to the probability that a system in the state  $i$  at time  $s$  will be in the state  $j$  at time  $t$ , having made  $n$  transitions during the interval  $[s, t]$ :

$$p_{ji}^{(n)}(t, s) := \sum_k \int_s^t \exp \left[ - \int_u^t t_j(v) dv \right] \Pi_{jk}(u) t_k(u) p_{ki}^{(n-1)}(u, s) du. \quad (29)$$

Reading (29) from right to left, it is the probability that the system starting in state  $i$  at time  $s$ , has arrived at state  $k$  by time  $u$  having made  $n - 1$  transitions, multiplied by the probability that it makes a transition during  $[u, u + du]$ , multiplied by the probability that this transition is to the state  $j$ , multiplied by the probability that it remains in state  $j$  until  $t$ , integrated over all intermediate times,  $u$ , and summed over all intermediate states  $k$ .

Feller defines the finite-time transition functions of the ‘minimal’ process to be

$$p_{ji}(t, s) := \sum_{n=0}^{\infty} p_{ji}^{(n)}(t, s). \quad (30)$$

He then shows that  $p_{ji}(t, s)$  is a solution of both the forward and the backward Kolmogorov equations with all the stated properties, including satisfaction of the Chapman–Kolmogorov equations.<sup>7</sup>

One can further extend Feller’s theorem, at least in the case of finite  $I$ , to cover infinitesimal parameters of the form  $t_{ji}(t) = \Pi_{ji}(t)t_i(t)$  where  $t_i(t)$  has singularities, as long as these singularities are isolated and integrable (Bacciagaluppi, 1996a, Ch. 7, Appendix 2). However, *non-integrable* singularities also occur, in fact very naturally, namely whenever a probability  $p_i(t)$  vanishes. If  $p_i(t) = 0$ , then if  $p_i(s) \neq 0$  for some  $s < t$ , one must also have  $p_{ii}(t, s) = 0$ . But now, by (16), in order for this probability to vanish,  $t_i$  must have a non-integrable singularity at  $t$ . Thus, a more specific discussion of existence and uniqueness is needed in the general case.

In the next section, we shall show how to derive infinitesimal parameters for the evolution of the complete state of a system, as defined earlier. If the corresponding Kolmogorov equations can be uniquely solved, this will lead to Markov processes governing the evolution of the complete state. We cannot always prove as yet the existence of such a Markov process because of the above difficulties. On the other hand, as we shall briefly discuss, in the context of the Bohm theory, an analogous problem has been successfully treated already, which makes us expect that similar existence results hold also in the present case.

### 3 Framework for a Modal Dynamics

#### 3.1 Continuity of the Evolution of Spectral Projections

It is clear that the set  $S(t)$  is in general genuinely time-dependent. For example, in the Vermaas–Dieks interpretation, it is the set of spectral projections of the statistical operator,  $W(t)$ . It is this time-dependence of  $S(t)$  that complicates the discussion of transition probabilities.

Indeed,  $S(t)$  is time-dependent in two ways. First, the *cardinality* of  $S(t)$  can change in time. For example, let

$$W(t) = \cos^2(\theta t)P_1 + \sin^2(\theta t)P_2 \quad (31)$$

for some  $\theta$ , and some projections,  $P_1 \perp P_2$ . For most times,  $W(t)$  has two spectral projections,  $P_1$  and  $P_2$ , so that for these times  $S(t)$  contains two projections. However,

when  $\theta t = n\pi/4$  for any integer  $n$ ,  $W(t)$  has just one spectral projection, so that for these times,  $S(t)$  contains just one projection.

The second kind of time-dependence involves the time-dependence of the projections in  $S(t)$ . Again a simple example is helpful. Consider a case where a system evolves freely. That is, for  $W(0) = \sum_i w_i P_i(0)$ , we have

$$W(t) = U(t)W(0)U^\dagger(t) = \sum_i w_i U(t)P_i(0)U^\dagger(t), \quad (32)$$

where  $U(t)$  is the unitary group generated by the (time-independent) Hamiltonian of the free system.<sup>8</sup> In this case, the cardinality of  $S(t)$  is constant, but its elements are genuinely time-dependent.

How are we to deal with these forms of time-dependence? The first presents a particularly difficult problem, because our aim is to take as the state space of a stochastic process an index set,  $I$ , labeling the elements of  $S(t)$ . But, the theory of stochastic processes generally assumes that the state space is constant. The time-dependence of the elements of  $S(t)$  presents a slightly less serious problem. Suppose that we have solved the first problem, *i.e.*, that the cardinality of  $S(t)$  is indeed constant in time, so that we have a genuine state space,  $I$ . Then we need, for each time,  $t$ , some way to associate each element of  $S(t)$  with some value in  $I$ . Of course, we could simply assume the existence of a family of bijective maps,  $\mu_t : I \rightarrow S(t)$ , but it would be nice if instead we could exhibit them.

To begin to resolve these problems, it is useful to discover some general facts about how the spectral projections of a statistical operator evolve. Here we show that, in the finite-dimensional case, they evolve continuously. We also give a brief qualitative discussion of the infinite-dimensional case. (For a more detailed treatment, see Bacciagaluppi, Donald and Vermaas (1995).) Continuity makes easy a resolution of the problems that arise from the genuine time-dependence of  $S(t)$  as it is presently defined (for the atomic version).

We consider a system at a time,  $t = 0$ , with statistical operator  $W(0)$ .  $W(0)$  has a spectral resolution

$$W(0) = \sum_i w_i(0)P_i(0). \quad (33)$$

At a time  $t$  in a neighborhood of 0, the system's state will have a spectral resolution

$$W(t) = \sum_j w_j(t)P_j(t). \quad (34)$$

We are interested in the relation between these two spectral resolutions.

Notice that the evolution of  $W(t)$  is continuous, *i.e.*, for  $t$  sufficiently small,  $W(t)$  is close to  $W(0)$  (in the trace-class norm, which is the physically relevant norm for statistical operators). This fact by itself implies nothing for the behavior of the spectral projections of  $W(t)$  in a neighborhood of 0. However, it suggests that we consider  $W(t)$  as a perturbation of the operator  $W(0)$  (induced by the time evolution), and apply results from perturbation theory in order to compare the eigenprojections of the perturbed and unperturbed operators. In particular, we can use Rellich's theorem (see for instance Kato (1966), Rellich (1969), and Reed and Simon (1978)), the central theorem of analytic perturbation theory.

In the finite-dimensional case, Rellich's theorem states the following. Take a family of operators,  $A(z)$ , depending on a *complex* parameter  $z$ , and such that  $A(z)$

is self-adjoint if  $z$  is real. If the dependence of  $A(z)$  on  $z$  is (complex-)analytic in a neighborhood of a real  $z_0$ , then both the eigenvalues and the eigenprojections of  $A(z)$  are also analytic functions of  $z$  in the following sense (the assumption cannot be weakened to real analyticity, *i.e.*, infinite real differentiability).

If the eigenvalues of  $A(z)$  are distinct throughout a neighborhood of  $z_0$ , the theorem simply means that the corresponding mutually orthogonal eigenprojections  $P_i(z)$  will be analytic functions of  $z$  in a neighborhood of  $z_0$ . Instead, if the eigenvalues of  $A(z)$  cross in  $z_0$ , there is a one-to-many correspondence between the eigenprojections of  $A(z)$  at  $z = z_0$  and  $z \neq z_0$ . However, the trajectories  $P_i(z)$ , which are analytic for  $z \neq z_0$ , can be *analytically extended* also to  $z = z_0$ .

Thus, if the statistical operators  $W(t)$  (extended to a complex neighborhood of the real line) depend complex-analytically on  $t$ , then the eigenprojections  $P_i(t)$  will be analytic functions of  $t$  with none but *removable* singularities, situated at the points in which the eigenvalues of  $W(t)$  cross. At such points, following Bacciagaluppi, Donald and Vermaas (1995), we propose to define the definite properties of the system as the corresponding limits of the  $P_i(t)$ . This proposal is a dynamically motivated *extension* of the modal interpretation. (See also the discussion by Bacciagaluppi (1996a, Ch. 6).) To put it differently, we postulate that (for an atomic system)  $S(t)$  is not always given by the spectral resolution of  $W(t)$ , but by the set of projections that are the analytic continuations of the eigenprojections of  $W(t)$  when the eigenvalues cross.

A particularly simple example is afforded by (31). We noted before that in that case,  $\tilde{S}(t)$  contains two elements except when  $\theta t = n\pi/4$  for any integer  $n$ . However, these singularities are clearly isolated, and we can postulate that for times,  $t$ , such that  $\theta t = n\pi/4$  for some integer  $n$ ,  $S(t) := \{P_1, P_2\}$ . This modification of the definition of  $S(t)$  clearly makes the projections in  $S(t)$  evolve analytically. (Indeed, they are constant.)

We thus only have to show that if  $W(t)$  is a statistical operator, then it is an analytic function of  $t$ . To do so, it is sufficient to consider the so-called ‘reduced’ states, *i.e.*, those obtained by partial tracing over the state of some ‘compound’ system. (We assume that the state of the universe is pure, so that its spectral resolution obviously evolves continuously.) Let the Hilbert space for the subsystem of interest be  $\mathcal{H}^\alpha$ , and let the Hilbert space for the rest of the compound system (ultimately, the rest of the universe) be  $\mathcal{H}^\beta$ . The Hilbert space for the compound system is therefore  $\mathcal{H} = \mathcal{H}^\alpha \otimes \mathcal{H}^\beta$ . We distinguish three cases, depending on the dimensionalities of the different Hilbert spaces: (1) both  $\mathcal{H}^\alpha$  and  $\mathcal{H}^\beta$  are finite-dimensional; (2)  $\mathcal{H}^\alpha$  is finite-dimensional, but  $\mathcal{H}^\beta$  is infinite-dimensional; (3)  $\mathcal{H}^\alpha$  is infinite-dimensional. Here we shall treat explicitly only the elementary case (1), with some brief remarks on cases (2) and (3).

We assume throughout that the total system is isolated and evolves according to the Schrödinger equation with a constant Hamiltonian,  $H$ :

$$|\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle, \quad (35)$$

given the initial state  $|\Psi(0)\rangle$ . Let  $E_1, \dots, E_N$  and  $|e_1\rangle, \dots, |e_N\rangle$  be the eigenvalues and eigenvectors of  $H$ , and write

$$|\Psi(0)\rangle = \sum_j \lambda_j |e_j\rangle. \quad (36)$$

Then

$$|\Psi(t)\rangle = \sum_j \lambda_j e^{-iE_j t} |e_j\rangle. \quad (37)$$

Introduce any product basis in  $\mathcal{H} = \mathcal{H}^\alpha \otimes \mathcal{H}^\beta$ , with basis vectors, say,  $|\psi_m^\alpha\rangle \otimes |\psi_n^\beta\rangle$ . Then, the reduced state of  $\alpha$  is given by

$$W^\alpha(t) = \sum_{i,j,k} \lambda_j \bar{\lambda}_k e^{-i(E_j - E_k)t} \langle \psi_i^\beta | e_j \rangle \langle e_k | \psi_i^\beta \rangle \quad (38)$$

(where the overbar denotes complex conjugation). Because every  $|e_j\rangle$  is some (finite) linear combination of the  $|\psi_m^\alpha\rangle \otimes |\psi_n^\beta\rangle$ ,

$$|e_j\rangle = \sum_{m,n} c_{mn}^j |\psi_m^\alpha\rangle \otimes |\psi_n^\beta\rangle, \quad (39)$$

the matrix elements of  $W(t)$  will be (finite) linear combinations of the functions  $e^{-i(E_j - E_k)t}$ , and thus analytic in  $t$ . This result is what we needed to establish.

For case (2), it can be shown (using estimates on the trace-class norm of  $W^\alpha(t)$ ) that if  $|\Psi(t)\rangle$  in (35) depends analytically on  $t$ , then  $W^\alpha(t)$  also depends analytically on  $t$ . Analyticity of  $|\Psi(t)\rangle$  is assured if one restricts the initial state (36) to a certain dense set in  $\mathcal{H}$ . Finally, in case (3), Rellich's theorem holds also in infinite dimensions for eigenvalues of the unperturbed operator that are isolated and of finite multiplicity. All nonzero eigenvalues of  $W^\alpha(t)$  are isolated and of finite multiplicity. Failure of the theorem for the eigenvalue 0 means that trajectories can be born from or can die into the null space of  $W^\alpha(t)$ . A fuller discussion of cases (2) and (3) is given in Bacciagaluppi, Donald, and Vermaas (1995). In the following, we shall assume that  $\mathcal{H}^\alpha$  is finite-dimensional and assume analyticity of  $W(t)$ .

Using these results, then, the two complications mentioned earlier are resolved. In particular, using the extended atomic version of the modal interpretation, the cardinality of the set  $S(t)$  is time-independent, and therefore can be indexed by a fixed index set,  $I$ , at all times. Moreover, continuity makes easy the definition of the family of functions  $\mu_t$ . At some initial time,  $t_0$ , arbitrarily associate with each element of  $S(t_0)$  some element of the index set,  $I$ . That is, choose  $\mu_{t_0}$  arbitrarily. For all later times,  $\mu_t$  is determined by the rule  $\mu_t(i) :=$  the (continuous) evolute of  $\mu_{t_0}(i)$  for all  $i$  and all  $t > t_0$ .

Using this definition of the maps  $\mu_t$ , we can focus attention on finding transition probabilities for a stochastic process,  $V_t$ , whose state space is  $I$ . With the help of the map  $\mu_t$ , such a process induces a stochastic evolution on the elements of  $S(t)$  (as defined by the extended atomic version), which, as we saw, will itself induce a stochastic evolution of the complete ‘modal’ state of a physical system (given by the ultrafilters,  $m$ , as defined in Section 1.1).

As important as these results on continuity are, we note that in *principle*, the problems that we have mentioned can be solved even if continuity fails. If the cardinality of  $S(t)$  as originally given changes in time, then find the  $S(t)$  with the greatest cardinality and decompose the elements of  $S(t)$  at the other times into ‘fiduciary’ elements. For example, suppose that for some  $t$ ,  $S(t)$  has the greatest cardinality,  $N$ , while  $S(s)$  (for some  $s \neq t$ ) has cardinality  $N - 2$ . In that case, there must be some multi-dimensional projections in the faux-Boolean algebra generated by  $S(s)$ . Suppose for illustration that  $S(s)$  contains a three-dimensional element,  $P$ . Then

we can decompose  $P$  into three one-dimensional, mutually orthogonal, projections,  $P = P_1 + P_2 + P_3$ , replacing  $P$  in  $S(s)$  with these three projections. The resulting faux-Boolean algebra will contain the old one as a subalgebra. Hence any dynamics involving the new algebra will induce a dynamics on the old algebra. And obviously the maps  $\mu_t$  can be defined arbitrarily.

However, we also emphasize that the existence in principle of a solution to the two problems is far less comforting than the actual existence provided by the results on analyticity discussed above. Moreover, as we will see, at least some reasonable dynamics require that the elements of  $S(t)$  be differentiable in time, and without the results on analyticity, there is no guarantee of differentiability. In any case, henceforth we ignore these details, and concentrate on finding  $V_t$ .

### 3.2 Infinitesimal Parameters and Single-Time Probabilities

We must now show how proceeding from a definition of the infinitesimal parameters to the finite-time transition probabilities (as in Theorems 1 and 2), we can ensure that our finite-time transition probabilities are consistent with the quantum-mechanical single-time probabilities. We do so in this section.

But first: a remark about notation. Later we will be considering compound systems, and we will be interested in their single-time joint probabilities as well as their joint transition probabilities. The former we denote by

$$p_{i,j,\dots}(t) := p(\alpha \text{ has } i \text{ at } t, \beta \text{ has } j \text{ at } t, \dots).$$

(Commas separate the indices of different subsystems.) The latter we denote by

$$p_{jn;im}(t, s) := p(\alpha \text{ has } j \text{ at } t, \beta \text{ has } n \text{ at } t, \dots | \alpha \text{ has } i \text{ at } s, \beta \text{ has } m \text{ at } s, \dots).$$

(A semicolon separates the later-time indices from the earlier-time indices.) However, we continue to denote transition probabilities for a single system by  $p_{ji}(t, s)$ . (These conventions hold also for the infinitesimal parameters.)

*From the definition of  $p_{ji}(t, s)$  and  $p_i(t)$ , it is evident that*

$$p_j(t) = \sum_i p_{ji}(t, s)p_i(s) \tag{40}$$

and

$$p_j(s) = \sum_i p_{ij}(t, s)p_j(s). \tag{41}$$

Therefore, we arrive at the difference equation

$$p_j(t) - p_j(s) = \sum_i p_{ji}(t, s)p_i(s) - p_{ij}(t, s)p_j(s). \tag{42}$$

Using (42), we can write

$$p_j(t + \varepsilon) - p_j(t) = \sum_i p_{ji}(t + \varepsilon, t)p_i(t) - p_{ij}(t + \varepsilon, t)p_j(t). \tag{43}$$

Divide both sides of (43) by  $\varepsilon$  and take the limit  $\varepsilon \rightarrow 0$ . Assuming that the limits exist (*i.e.*, that the  $t_{ji}(t)$  are well-defined), we find that  $p_j(t)$  is differentiable and that

$$\dot{p}_j(t) = \sum_i [t_{ji}(t)p_i(t) - t_{ij}(t)p_j(t)]. \quad (44)$$

Equation (44) is a standard master equation. (In our case, it will turn out that  $t_{ji}(t)$  is singular only when  $p_i(t) = 0$ , so that with the convention  $\infty \times 0 = 0$ , (44) holds even when  $t_{ji}(t)$  is singular.) Notice that we nowhere assumed the Markov property in this derivation of the master equation. That is, the  $t_{ji}(t)$  could be the infinitesimal parameters of a non-Markov process. On the other hand, we have seen that, given  $t_{ji}(t)$ , one can construct a canonical Markov process that has the  $t_{ji}(t)$  as infinitesimal parameters, and it is this route to finite-time transition probabilities that we follow. (There can be no construction of a unique non-Markov process from its infinitesimal parameters because these cannot carry information about how the *history* of the process affects the transition probabilities.)

The master equation (44) is the relation we need between single-time distributions and infinitesimal parameters. We have now to investigate how one can go about solving the master equation for the  $t_{ji}(t)$ , *given* a set of single-time distributions  $p_j(t)$  and their derivatives. We shall follow the strategy used by Bell (1984). Like Bell, from now on we deal exclusively with the case of a *finite* index set  $I$ . (Hence Theorems 1 and 2, with the above-mentioned limitations, and the results of Section 3.1 apply.) In fact, our discussion of stochastic processes clarifies the background and possible limitations also of the work of Bell (1984), Vink (1993) and Bub (1996, 1997).

Define a *probability current*  $j_{ji}(t)$  between trajectories, representing the ‘net flow’ of probability from  $i$  to  $j$ :

$$j_{ji} := t_{ji}p_i - t_{ij}p_j \quad (45)$$

(where we have suppressed the argument  $t$ , as we will often do henceforth). The definition of  $j_{ji}$  implies

$$j_{ij} = -j_{ji}. \quad (46)$$

(That is, the  $j_{ji}$  form an antisymmetric matrix.)

Further, equation (44) implies a *continuity equation* for the current:

$$\dot{p}_j = \sum_i j_{ji}. \quad (47)$$

A solution to (44) can thus be obtained by first finding a current  $j_{ji}$  that satisfies (46) and (47), and then, with this  $j_{ji}$ , finding functions  $t_{ji}$  that satisfy (45). By Theorems 1 and 2, at least if the  $t_{ji}$  are continuous and non-singular, one can then construct an essentially unique stochastic process describing the evolution of the system.

Later, we shall exhibit some explicit expressions for  $j_{ji}$ . For now, we consider how to solve (45), with the added constraint that the  $t_{ji}$  be a set of (finite-valued) infinitesimal parameters. That is, we consider how to solve (45) under the constraints

$$t_{ji} \geq 0 \quad (j \neq i), \quad (48)$$

and

$$\sum_j t_{ji} = 0. \quad (49)$$

The latter requirement (which is equation (13) again) defines  $t_{ii}$  in terms of the  $t_{ji}$  for  $j \neq i$ . In fact, (45) is a system of equations for the  $t_{ji}$  with  $j \neq i$ , the equations being vacuous for  $j = i$ . Therefore, we only need to solve (45) under the constraint that the  $t_{ji}$  be non-negative when  $i \neq j$ .

From (45) one has (assuming  $p_j > 0$ )

$$t_{ij} = \frac{t_{ji}p_i - j_{ji}}{p_j}. \quad (50)$$

By (48),  $t_{ij}$  must be positive, and therefore we need  $t_{ji}p_i - j_{ji} \geq 0$ , or (for  $p_i > 0$ )

$$t_{ji} \geq \frac{j_{ji}}{p_i}, \quad (51)$$

whence, again by (48),

$$t_{ji} \geq \max \left\{ 0, \frac{j_{ji}}{p_i} \right\}. \quad (52)$$

From Equations (48)–(52) above, it follows that the general solution to (45), given  $j_{ji}$ , can be found by choosing for every pair  $j < i$  a function  $t_{ji}$  that satisfies (52) (and which can be chosen to be continuous). The  $t_{ij}$  are then uniquely determined by (50), and the  $t_{ii}$  by (49).

The most natural choice for a solution to (45) seems therefore to be the following: for  $j < i$  choose

$$t_{ji} := \max \left\{ 0, \frac{j_{ji}}{p_i} \right\}. \quad (53)$$

One easily checks with (50) that in this case also

$$t_{ij} = \max \left\{ 0, \frac{j_{ij}}{p_j} \right\}. \quad (54)$$

Thus,  $t_{ji}$  is given by (53) for all  $j \neq i$ . In fact, this is the choice made by Bell (1984) for the solution of (45).

Bell's choice is clearly motivated by the analogy with the guiding condition in Bohm's (1952) theory. Vink (1993) discusses how, in the appropriate sense, the Bohm theory is in fact the continuum limit of a dynamics of this kind, and how different solutions to (45) lead to different kinds of theories, one example being Nelson's (1985) stochastic mechanics.<sup>9</sup>

On the other hand, it is also clear from (45) that whenever  $p_i = 0$  and  $j_{ji} > 0$ , the infinitesimal parameters are singular. It is actually possible to have  $j_{ji} = 0$  whenever  $p_i = 0$  (the current we shall construct has this property). However, as we have seen discussing Theorem 2, singularities will arise anyway.<sup>10</sup>

We have now a canonical procedure for constructing Markov processes with given (differentiable) single-time distributions  $p_j$ . We solve the linear system of equations

(47) for the  $j_{ji}$  under the additional constraint (46). We then define infinitesimal parameters  $t_{ji}$  by (53). If these are continuous and regular enough, then by Theorems 1 and 2 we can construct a Markov process having the  $t_{ji}$  as its infinitesimal parameters. (In the singular case, these theorems are not enough. See the remarks in Section 4.3.) One can also see that this Markov process will have the given  $p_j$  as its single-time distributions, by looking at the continuity equation (47) from the inverse perspective, now as a system of linear differential equations for the  $p_j$  given the  $j_{ji}$ . By the uniqueness theorems for systems of ordinary linear differential equations, if two stochastic processes (Markovian or non-Markovian) share the same infinitesimal parameters and the same initial distribution  $p_j(0)$ , then they will have the same single-time distributions  $p_j(t)$  for all  $t$ .

It is clear now that the problem of finding a dynamics, even a Markovian one, is vastly *underdetermined*. Even if we limit ourselves to Bell's choice (53) in the solution to (45), the continuity equation (47) has infinitely many solutions that satisfy (46). It is also clear that the requirement of continuity of the  $t_{ji}$  will not force one particular choice of current. Our major remaining task is therefore to choose an appropriate solution to the continuity equation (47).

Also this situation has a parallel in the Bohm theory. Ghirardi and Deotto (1997) have recently shown that it is possible to construct infinitely many deterministic Bohm-like theories that yield the correct single-time position distribution, simply by adding a divergence-free term to the standard choice for the probability current satisfying the continuity equation

$$\frac{d}{dt}\rho(x,t) = \nabla \cdot j(x,t). \quad (55)$$

An example of such a Ghirardi–Deotto current can be seen in Bohm and Hiley's treatment of the Pauli equation as the non-relativistic limit of the Dirac equation (Bohm and Hiley, 1993, section 10.4), where the current obtained differs by a divergence-free term from the current used in the purely non-relativistic treatment of the Pauli equation (*ibid.*, section 10.2).

## 4 An Explicit Dynamics

### 4.1 The ‘Minimal Flow’ Current

We shall be concentrating on a current that we call the ‘generalized Schrödinger current’, and the dynamics that it induces via (53). But first, we exhibit one other solution, mainly because it applies to versions other than the atomic version, if they are, indeed, feasible. (The generalized Schrödinger current is definable only for the atomic version.)

As we have noted, both (45) and (47) are linear systems of equations, to be solved under certain constraints. The case of (47) is very straightforward: the constraints are given by the additional equations (46), which are also linear. The system consisting

of (46) and (47) has infinitely many solutions. One possible selection criterion would be to minimise the overall flow of probability, in the sense that, say,

$$\sum_{i,j} j_{ji}^2 \quad (56)$$

be minimised. This criterion is mathematically very tractable (in the finite case we are now studying), and leads to the following current:

$$j_{ji} = \frac{1}{D}(\dot{p}_j - \dot{p}_i), \quad (57)$$

where  $D$  is the number of elements of  $S$ . (We omit the proof: see Bacciagaluppi (1996a, Ch. 7, Appendix 1).)

Note that this current can be applied whenever we have well-defined joint probabilities for the universe, that is, joint probabilities covering the joint possession of properties by all subsystems. For example, in the non-atomic version with a preferred factorization given by  $\mathcal{H}_{\text{univ}} = \mathcal{H}^\alpha \otimes \dots \otimes \mathcal{H}^\omega$ , we would need joint probabilities covering all atomic systems *plus* all combinations of atomic systems. In a universe with only two atomic subsystems,  $\alpha$  and  $\beta$ , we could write these joint probabilities as

$$p_{i_\alpha, i_\beta, i_{\alpha\beta}}(t). \quad (58)$$

We would then define the ‘universal current’ to be

$$j_{j_\alpha j_\beta j_{\alpha\beta}; i_\alpha i_\beta i_{\alpha\beta}} = \frac{1}{D}(\dot{p}_{j_\alpha, j_\beta, j_{\alpha\beta}} - \dot{p}_{i_\alpha, i_\beta, i_{\alpha\beta}}), \quad (59)$$

where  $D$  is now the number of elements in the set of all *joint* states. Whenever we have differentiable universal joint probabilities we can, in principle at least, define a dynamics in this way.

However, such a dynamics has a number of less pleasing properties that motivate us to seek a more ‘quantum-mechanical’ current. For example, it is clear that ‘minimal flow’ for the universal current does *not* entail minimal flow for the currents on subsystems induced by the universal current. Moreover, as we will discuss in detail later, there is a standard expression in quantum mechanics for the current for a time-*independent* set of eigenprojections, and the minimal flow current does not reduce to this standard current in the case where  $S(t)$  happens to be time-independent.

## 4.2 Deterministic Evolution for Free Systems

Before we move on to the task of finding a more satisfactory current, we pause to discuss a result first due to Vermaas (1996). It concerns the evolution of the possessed properties of a freely evolving system. Under certain fairly natural constraints, Vermaas shows that freely evolving systems must evolve deterministically. This result

does limit the range of acceptable currents (though there seem to remain infinitely many possibilities). In this section, we give a slightly different (and in some ways more limited) justification for the same result, mainly because we find the example we shall use to be instructive.

We now consider a motivation for the principle that freely evolving systems evolve deterministically. Our motivation is not really a ‘proof’, but it is instructive nonetheless.<sup>11</sup>

Consider a system in the initial state

$$|\Psi(0)\rangle = |\alpha_1\rangle \otimes |\beta_1\rangle, \quad (60)$$

and let its evolution operator be

$$\begin{aligned} U^\alpha \otimes U^\beta &= \\ &= \left( \cos \omega t |\alpha_1\rangle\langle\alpha_1| + i \sin \omega t |\alpha_2\rangle\langle\alpha_1| + \cos \omega t |\alpha_2\rangle\langle\alpha_2| + i \sin \omega t |\alpha_1\rangle\langle\alpha_2| \right) \otimes \mathbb{I}^\beta, \end{aligned} \quad (61)$$

where  $\mathbb{I}^\beta$  is the identity operator on  $\mathcal{H}^\beta$ . (This operator is unitary.) The system  $\alpha$  then has its own state vector,  $|\alpha_1\rangle$ , so that  $\alpha$  definitely possesses  $P_1^\alpha$  ( $= |\alpha_1\rangle\langle\alpha_1|$ ) at time  $t = 0$ . The evolution operator acting on  $|\Psi(0)\rangle$  yields

$$|\Psi(t)\rangle = (\cos \omega t |\alpha_1\rangle + i \sin \omega t |\alpha_2\rangle) \otimes |\beta_1\rangle, \quad (62)$$

so that  $\alpha$  continues to have its own state vector,  $\cos \omega t |\alpha_1\rangle + i \sin \omega t |\alpha_2\rangle$ , which goes smoothly from  $|\alpha_1\rangle$  to  $|\alpha_2\rangle$  and back again. Hence  $\alpha$  ‘follows’ this smooth transition. We might say that in this case the evolution of the *complete* state of  $\alpha$  follows the Hamiltonian evolution. The same holds if the initial state of the system is instead  $|\Psi(0)\rangle = |\alpha_2\rangle \otimes |\beta_2\rangle$ . In both cases, the evolution is necessarily deterministic.

Now, suppose instead that the initial state of the system is

$$|\Psi(0)\rangle = c_1 |\alpha_1\rangle \otimes |\beta_1\rangle + c_2 |\alpha_2\rangle \otimes |\beta_2\rangle, \quad (63)$$

but with the same evolution operator, (61). In this case,  $\alpha$  initially possesses either  $P_1^\alpha$  or  $P_2^\alpha$ . Indeed, the reduced statistical operator for  $\alpha$  is

$$\begin{aligned} W^\alpha(t) &= |c_1|^2 \left( \cos^2 \omega t |\alpha_1\rangle\langle\alpha_1| + i \sin \omega t \cos \omega t |\alpha_2\rangle\langle\alpha_1| \right. \\ &\quad \left. - i \sin \omega t \cos \omega t |\alpha_1\rangle\langle\alpha_2| + \sin^2 \omega t |\alpha_2\rangle\langle\alpha_2| \right) \\ &\quad + |c_2|^2 \left( \sin^2 \omega t |\alpha_1\rangle\langle\alpha_1| - i \sin \omega t \cos \omega t |\alpha_2\rangle\langle\alpha_1| \right. \\ &\quad \left. + i \sin \omega t \cos \omega t |\alpha_1\rangle\langle\alpha_2| + \cos^2 \omega t |\alpha_2\rangle\langle\alpha_2| \right) = \\ &=: |c_1|^2 P'_1(t) + |c_2|^2 P'_2(t). \end{aligned} \quad (64)$$

As one would expect,  $P'_1(t)$  goes smoothly from  $P_1^\alpha$  to  $P_2^\alpha$  and back again as  $t$  goes from 0 to  $\pi/2\omega$ , and conversely for  $P'_2(t)$ .

Knowing that the Hamiltonian takes  $|\alpha_1\rangle$  smoothly to  $|\alpha_2\rangle$  and vice versa, it is natural to suppose that here too, the evolution of the complete state follows the Hamiltonian evolution. That is, it is natural to suppose that if the system possesses  $P_1^\alpha$  initially, then it will at all times possess  $P'_1(t)$ , and similarly for  $P_2^\alpha$  and  $P'_2(t)$ . Such an evolution is deterministic.

This argument can be repeated for any system that evolves freely. One can always ask what the system would do if it were in a pure state (corresponding to one of the elements in its spectral resolution, ignoring subtleties involving degeneracy, which do not really affect the argument). The answer will be that the complete state *must* evolve deterministically. But then it seems reasonable to impose this deterministic evolution of the complete state back onto the system when its state is not pure. Hence we shall assume that the complete state of any freely evolving system itself evolves deterministically, ‘following’ the Hamiltonian evolution. (We emphasize again that Vermaas (1996) *has* derived this result from some fairly natural conditions.)

#### 4.3 The Generalized Schrödinger Current

The motivation of the previous subsection was derived partly from a desire that the possessed properties of systems evolve in a recognizably ‘quantum-mechanical’ way. But the minimal flow current is not at all recognizably ‘quantum-mechanical’. Nor does it yield deterministic evolution for all freely evolving systems. In this section, we propose a current that is more quantum-mechanical and satisfies stability.

As a matter of fact, there is a fairly standard way of deriving a probability current from the Schrödinger equation. (See, *e.g.*, Cohen-Tannoudji *et al.* (1977, pp. 238–240).) This kind of current is what is used in the Bohm theory to guide the motion of particles. In Bell’s (1984) stochastic modification of the Bohm theory, it is used to derive transition probabilities between the eigenprojections of the privileged observable (‘beable’) of the theory, supposed to be fermion number density. Vink (1993) has used Bell’s treatment to derive a probability current between eigenprojections of any given observable, and Bub (1996, 1997) uses Bell’s and Vink’s current, which we call the ‘Schrödinger current’, to define a dynamics for his own interpretation of quantum theory.

We cannot use the Schrödinger current in its usual form, because in the atomic version, the privileged observable (defined from  $S(t)$ —see the discussion in Section 1.2) is genuinely time-dependent. However, we shall generalize the Schrödinger current to the case of a time-dependent privileged observable. (Hence, as we said in Section 1.2, our work here is a generalization of the work by Bell, Vink and Bub.) We proceed by defining a current for general composite systems—and in principle, the universe. This current will give us a dynamics for the joint state of the universe, which induces a dynamics for any subsystem.

Recall from Section 1.3 that the state space for the universe is given by (2) (and for us, the universe has a finite-dimensional Hilbert space given by  $\mathcal{H}_{\text{univ}} = \mathcal{H}^\alpha \otimes \dots \otimes \mathcal{H}^\omega$ , so that this state space is finite). The single-time joint probability density is given by

$$p_{i_\alpha, \dots, i_\omega}(t) := \langle \psi(t) | P_{i_\alpha}^\alpha(t) \otimes \dots \otimes P_{i_\omega}^\omega(t) | \psi(t) \rangle. \quad (65)$$

For convenience, we will usually denote the (ordered) set of indices  $(i_\alpha, \dots, i_\omega)$  by a collective index,  $i$ , so that (65) can be written

$$p_i(t) = \langle \psi(t) | P_i(t) | \psi(t) \rangle. \quad (66)$$

Let us assume first that each  $P_i$  is one-dimensional and time-independent, given by  $|r_i\rangle\langle r_i|$  (so that  $\{|r_i\rangle\}$  is an orthonormal set, and could be considered the set of eigenvectors of the preferred observable,  $R$ , in Bub's interpretation). Then from the Schrödinger equation, it follows that

$$\begin{aligned} \dot{p}_i(t) &= -i\langle \psi(t) | r_i \rangle \langle r_i | H | \psi(t) \rangle + i\langle r_i | \psi(t) \rangle \langle \psi(t) | H^\dagger | r_i \rangle = \\ &= 2\text{Im}\left[\langle \psi(t) | r_i \rangle \langle r_i | H | \psi(t) \rangle\right]. \end{aligned} \quad (67)$$

We must now choose a current,  $j_{ji}(t)$ , that satisfies the continuity equation (47), where  $\dot{p}_j(t)$  is given by (67). The following simple choice is the ‘textbook’ expression:

$$j_{ji}(t) := 2\text{Im}\left[\langle \psi(t) | r_j \rangle \langle r_j | H | r_i \rangle \langle r_i | \psi(t) \rangle\right]. \quad (68)$$

This current is the one used by Bell and Vink (and, Vink (1993) argues, in a suitable limit it becomes the current of Bohm’s theory).

When the  $P_i$  are not one-dimensional, still each  $P_i$  is spanned by some set of vectors in  $\{|r_i\rangle\}$ , so that the density, time-derivative of the density, and current are obtained by summing (66), (67), and (68) over  $i$  and  $j$  appropriately:

$$p_i(t) = \langle \psi(t) | P_i | \psi(t) \rangle, \quad (69)$$

$$\dot{p}_i(t) = 2\text{Im}\left[\langle \psi(t) | P_i H | \psi(t) \rangle\right], \quad (70)$$

$$j_{ji}(t) = 2\text{Im}\left[\langle \psi(t) | P_j H P_i | \psi(t) \rangle\right]. \quad (71)$$

We now face the problem that, for us, the  $P_j$  are genuinely time-dependent. Hence in the density in (66), the  $P_i(t)$  are truly time-dependent, and (67) does *not* give the time-derivative of  $p_i(t)$ . Instead, it is

$$\dot{p}_i(t) = 2\text{Im}\left[\langle \psi(t) | P_i(t) H | \psi(t) \rangle\right] + \langle \psi(t) | \dot{P}_i(t) | \psi(t) \rangle. \quad (72)$$

We seek a current that satisfies the continuity equation (47) with the left-hand side given by (72).

Before we go on to find such a current, we note that the approach we are now following makes two assumptions not required by the ‘minimal flow’ dynamics of Section 4.1. First, it assumes that the ‘universal’ single-time joint probabilities are given by (66). In a non-atomic version, (66) does not give the single-time joint probabilities. Nor is there any obvious way to represent a universal joint state (one

property for each system) by any projection operator, as there is in the atomic version. Second, our present approach assumes that the  $P_i(t)$  are differentiable. By the results on analyticity that we discussed in Section 3.1, this assumption is acceptable.

One might wonder why we do not simply stick with the minimal flow dynamics, or some other dynamics derived in a similar way, when the present approach is so much less general. The answer is that we find the present approach to be much more closely tied to traditional methods and questions in quantum mechanics, and while such ties are no reason to rule out other approaches, they are sufficient to motivate our pursuing this approach. Indeed, we will now show that there is a very natural way to find a generalization of the Schrödinger current for time-dependent  $P_i(t)$ . (The following line of reasoning is due to James Cushing (p. c.).)

For each subsystem,  $\nu$ , we write  $P_{i_\nu}^\nu = |\nu_i\rangle\langle\nu_i|$ . We assume that the universe is in a pure state, and we write it in the basis given by the  $P_{i_\nu}^\nu$ , i.e., in the basis  $\{|\alpha_{i_\alpha}\rangle \otimes \dots \otimes |\omega_{i_\omega}\rangle\}$ :

$$|\Psi(t)\rangle = \sum_{i_\alpha, \dots, i_\omega} c_{i_\alpha, \dots, i_\omega}(t) |\alpha_{i_\alpha}(t)\rangle \otimes \dots \otimes |\omega_{i_\omega}(t)\rangle. \quad (73)$$

(If any of the  $P_{i_\nu}^\nu$  is more than one-dimensional, then we just choose a set,  $\{|\nu_k\rangle\}$ , such that each  $P_{i_\nu}^\nu$  is spanned by some subset of  $\{|\nu_k\rangle\}$ .) As before, we let a collective index,  $i$ , stand in for  $i_\alpha, \dots, i_\omega$ , so that we may write

$$|\Psi(t)\rangle = \sum_i c_i(t) |q_i(t)\rangle \quad (74)$$

for some set of (tensor-product) vectors,  $\{|q_i(t)\rangle\}$ . Similarly, we label the  $P_{i_\alpha}^\alpha(t) \otimes \dots \otimes P_{i_\omega}^\omega(t)$  with the collective index  $i$ .

At every time,  $t$ ,  $\{P_i(t)\}$  is a set of mutually orthogonal projections, so that there exists some family of unitary operators,  $\{O(t, s)\}$ , such that  $P_i(t) = O(t, s)P_i(s)O^\dagger(t, s)$  for all  $t$  and  $s$ . Write  $O(t) := O(t, 0)$ . We define:

$$|\Psi'(t)\rangle := O^\dagger(t)|\Psi(t)\rangle, \quad (75)$$

so that

$$|\Psi'(t)\rangle = \sum_i c_i(t) |q_i(0)\rangle. \quad (76)$$

Therefore, as far as the atomic version is concerned,  $|\Psi'(t)\rangle$  differs from  $|\Psi(t)\rangle$  only in the fact that its definite-valued projections are time-independent (and are, in fact, given by the definite-valued projections for  $|\Psi(t)\rangle$  at time  $t = 0$ ). The probabilities attached to these time-independent projections are the *same* as the probabilities attached to their time-dependent images under the map given by  $O(t)$ . But we already know how to write down a current for the time-independent case. So the obvious strategy is to write down the current for  $|\Psi'(t)\rangle$ , then translate the result *back* in terms of the time-dependent projections, again using the map given by  $O(t)$ . In this way, we will have derived in a very natural manner a current associated with  $|\Psi(t)\rangle$ , for the time-dependent set  $S(t)$ .

Using (75) and the Schrödinger equation, we have that

$$\begin{aligned} i \frac{\partial |\Psi'(t)\rangle}{\partial t} &= i\dot{O}^\dagger(t)|\Psi(t)\rangle + iO^\dagger(t)\frac{\partial |\Psi(t)\rangle}{\partial t} = \\ &= i\dot{O}^\dagger(t)O(t)|\Psi'(t)\rangle + O^\dagger(t)H(t)O(t)O^\dagger(t)|\Psi(t)\rangle. \end{aligned} \quad (77)$$

Defining

$$\tilde{H}(t) := i\dot{O}^\dagger(t)O(t) + O^\dagger(t)H(t)O(t), \quad (78)$$

we can therefore write

$$i \frac{\partial |\Psi'(t)\rangle}{\partial t} = \tilde{H}(t)|\Psi'(t)\rangle. \quad (79)$$

(The operator  $\tilde{H}(t)$  is indeed self-adjoint.) As we said,  $|\Psi'(t)\rangle$  gives rise to time-independent definite-valued projections, which are in fact just  $P_i(0)$ , so that the current given by (71) is applicable:

$$j_{ji}(t) = 2\text{Im}\left[\langle\Psi'(t)|P_j(0)\tilde{H}(t)P_i(0)|\Psi'(t)\rangle\right]. \quad (80)$$

Substituting in for  $\tilde{H}(t)$  and using the equality  $O(t)P_j(0)O^\dagger(t) = P_j(t)$ , we get:

$$j_{ji}(t) = 2\text{Im}\left[i\langle\Psi(t)|O(t)P_j(0)\dot{O}^\dagger(t)P_i(t)|\Psi(t)\rangle + \langle\Psi(t)|P_j(t)H(t)P_i(t)|\Psi(t)\rangle\right] \quad (81)$$

(where we have also used (75)). Now note that

$$\begin{aligned} \dot{P}_j(t) &= \dot{O}(t)P_j(0)O^\dagger(t) + O(t)P_j(0)\dot{O}^\dagger(t) = \\ &= \dot{O}(t)O^\dagger(t)P_j(t) + P_j(t)O(t)\dot{O}^\dagger(t), \end{aligned} \quad (82)$$

so that

$$O(t)P_j(0)\dot{O}^\dagger(t)P_i = \dot{P}_j(t)P_i(t) - \dot{O}(t)O^\dagger(t)P_j(t)P_i(t). \quad (83)$$

But the second term in the right-hand side of (83) is zero when  $i \neq j$  (and when  $i = j$  the entire current is zero anyway, as is clear already from (80)). Therefore (81) is

$$\begin{aligned} j_{ji}(t) &= 2\text{Im}\left[i\langle\Psi(t)|\dot{P}_j(t)P_i(t)|\Psi(t)\rangle + \langle\Psi(t)|P_j(t)H(t)P_i(t)|\Psi(t)\rangle\right] = \\ &= 2\text{Im}\left[\langle\Psi(t)|P_j(t)H(t)P_i(t)|\Psi(t)\rangle\right] \\ &\quad + \langle\Psi(t)|\dot{P}_j(t)P_i(t)|\Psi(t)\rangle + \langle\Psi(t)|P_i(t)\dot{P}_j(t)|\Psi(t)\rangle. \end{aligned} \quad (84)$$

Finally, because  $P_i(t)P_j(t) = 0$  for  $i \neq j$  we have

$$\frac{d}{dt} [P_i(t)P_j(t)] = \dot{P}_i(t)P_j(t) + P_i(t)\dot{P}_j(t) = 0, \quad (85)$$

so that  $P_i(t)\dot{P}_j(t) = -\dot{P}_i(t)P_j(t)$  and the current in (84) is therefore antisymmetric. Indeed, we can rewrite it as:

$$j_{ji}(t) = 2\text{Im}\left[\langle\Psi(t)|P_j(t)H(t)P_i(t)|\Psi(t)\rangle\right] + \langle\Psi(t)|(\dot{P}_j(t)P_i(t) - \dot{P}_i(t)P_j(t))|\Psi(t)\rangle. \quad (86)$$

The current in (86) is a natural generalization of the Schrödinger current. Note in particular that when  $\dot{P}_i(t) = \dot{P}_j(t) = 0$ , (86) reduces to the Schrödinger current of Bell and Vink. (Actually, only *one* of  $\dot{P}_i(t)$  or  $\dot{P}_j(t)$  need be zero.)

It is also useful to see explicitly that (86) satisfies the continuity equation (47). That it does so is clear from the fact that  $\sum_i \dot{P}_i(t) = 0$ , so that the extra term in the generalized current summed over  $i$  gives  $\langle\psi(t)|\dot{P}_j(t)|\psi(t)\rangle$ , which is just the extra term in  $\dot{p}_j(t)$  as given in (72).

We stress, however, that although the current in (86) is in some sense ‘natural’, it is certainly not the only current that generalizes the Schrödinger current. Indeed, to get a generalized Schrödinger current, we need only add to the Schrödinger current some antisymmetric term that, when summed over  $i$ , gives  $\langle\psi(t)|\dot{P}_j(t)|\psi(t)\rangle$ .<sup>12</sup> For example, the expression

$$\langle\psi(t)|\frac{1}{D}(\dot{P}(t) - \dot{P}_i(t))|\psi(t)\rangle, \quad (87)$$

which is reminiscent of the minimal flow dynamics, is also an adequate extra term (where  $D$  is the cardinality of  $S$ .)

In any case, using (86) we can now define infinitesimal parameters by (53). By the results of Section 3.1, we are allowed to assume analyticity of the probabilities  $p_i(t)$ . In fact, the  $p_i(t)$  will have only isolated zeros, and therefore the  $j_{ji}(t)/p_i(t)$  will be singular at most on a set of isolated points. Further, it is easy to see also that the  $j_{ji}(t)$  are analytic, because by the results of Section 3.1 the projections  $P_i(t)$  are. Therefore, the singularities of  $j_{ji}(t)/p_i(t)$  are just poles, and thus both  $j_{ji}(t)/p_i(t)$  and  $t_{ji}(t) = \max\{0, j_{ji}(t)/p_i(t)\}$  are continuous functions with values in  $[-\infty, \infty]$ .

Poles, however, are non-integrable singularities, and thus Theorems 1 and 2 only ensure that we can canonically construct a Markov process for the evolution of the possessed properties *between* the singularities. The question of the existence of *global* solutions is not yet settled. On the other hand, the situation is not surprising and possibly no great cause of concern, if one notes that the same problem arises also in the case of the Bohm theory, and that further, in the latter case, the existence and uniqueness of global solutions has been demonstrated under quite general conditions (Berndl *et al.* 1995).

To be precise, in the Bohm theory the guidance equation,

$$\frac{d}{dt}\mathbf{x}_i(t) = \frac{\mathbf{j}_i(\mathbf{x}_1, \dots, \mathbf{x}_N, t)}{\rho(\mathbf{x}_1, \dots, \mathbf{x}_N, t)}, \quad (88)$$

will become singular when  $\rho(\mathbf{x}_1, \dots, \mathbf{x}_N, t) = 0$ . However, Berndl *et al.* (1995) have shown that for a wide class of potentials the guidance equation admits unique global solutions for generic initial conditions. One can expect that for our choice of current and for Bell's choice of infinitesimal parameters, results similar to Berndl *et al.*'s will hold also in the modal interpretation.

As a final remark, we show that, as claimed before,  $p_i(t) = 0$  implies  $j_{ji}(t) = 0$ . Writing  $p_i(t) = \langle \psi(t) | P_i(t) | \psi(t) \rangle$  it becomes clear that  $p_i(t) = 0$  if and only if  $P_i(t) | \psi(t) \rangle = 0$ . In that case, (86) is

$$j_{ji}(t) = -\langle \psi(t) | \dot{P}_i(t) P_j(t) | \psi(t) \rangle. \quad (89)$$

And, because  $P_i(t) \dot{P}_j(t) = -\dot{P}_i(t) P_j(t)$ , (89) is

$$j_{ji}(t) = \langle \psi(t) | P_i(t) \dot{P}_j(t) | \psi(t) \rangle, \quad (90)$$

which is zero (under our assumption that  $P_i(t) | \psi(t) \rangle = 0$ ).

In this section, we have discussed the construction of a Markovian dynamics at the level of the *total* system for the evolution of the complete state in the atomic version (at least in the finite-dimensional case). The next section is devoted to the analysis of certain aspects of the (generally non-Markovian) dynamics induced on the atomic *subsystems*. In particular, we show that the choice (86) for the current leads to transition probabilities whose marginals are deterministic for freely evolving atomic systems.

## 5 Properties of the Generalized Schrödinger Dynamics

### 5.1 Determinism for Free Atomic Systems

Our discussion thus far leaves open the question of the properties of the dynamics of subsystems. Here we will begin to investigate these properties. In this subsection, we will be concerned with the dynamics of a freely evolving atomic system. We will show that such a system evolves deterministically, as discussed in Section 4.2. In the next subsection, we make a few remarks towards a general account of the dynamics of subsystems.

For simplicity of notation, we concentrate on the atomic system  $\alpha$ , and use  $i$  and  $j$  to label its states. The joint states of all of the other atomic systems in the universe we denote with collective indices  $m$  and  $n$ . To show that when  $\alpha$  evolves freely, it evolves deterministically, we must show that whenever the Hamiltonian for the universe takes the form  $H = H^\alpha \otimes \Pi^{\beta \dots \omega} + \Pi^\alpha \otimes H^{\beta \dots \omega}$ —i.e., whenever there are no interactions between  $\alpha$  and the rest of the universe—the infinitesimal parameters are such that for all  $n$  and for all  $i, m$  such that  $p_{im}(t) \neq 0$ ,  $t_{jn;im}(t) = 0$  whenever  $i \neq j$ .

One nice way to prove this claim is by using again the time-dependent transformation  $O(t)$  used in the derivation of the generalized Schrödinger current that we

gave in the previous section. Recall that there we defined the family of unitary operators,  $O(t)$ , to be such that  $P_i(t) = O(t)P_i(0)O^\dagger(t)$ , for every  $P_i(t) \in S(t)$ . When the Hamiltonian for the compound system is  $H = H^\alpha \otimes \mathbb{I}^{\beta \dots \omega} + \mathbb{I}^\alpha \otimes H^{\beta \dots \omega}$ ,  $\alpha$  evolves according to its *own* unitary group,

$$U(t) = e^{-iH^\alpha t}. \quad (91)$$

Hence the definite-valued projections for  $\alpha$  evolve according to  $P_i^\alpha(t) = U(t)P_i^\alpha(0)U^\dagger(t)$ . It follows that the definite-valued projections for the compound system, which may be denoted  $P_i^\alpha(t) \otimes P_m(t)$ , evolve according to

$$P_i^\alpha(t) \otimes P_m(t) = U(t) \otimes V(t)P_i^\alpha(0) \otimes P_m(0)U^\dagger(t) \otimes V^\dagger(t) \quad (92)$$

where  $\{V(t, s)\}$  (with  $V(t) := V(t, 0)$ ) is the appropriate family of unitary operators on the rest of the universe. [Note that  $V(t)$  bears no simple relation to the evolution operator on the rest of the universe, which is  $e^{-iH^{\beta \dots \omega}t}$ .]

We are going to substitute  $U(t) \otimes V(t)$  for  $O(t)$  in (78). To do so, we need to know its derivative as well. It is:

$$\begin{aligned} \frac{d}{dt}[U(t) \otimes V(t)] &= \frac{d}{dt}[e^{-iH^\alpha t} \otimes V(t)] = \\ &= -iH^\alpha U(t) \otimes V(t) + U(t) \otimes \dot{V}(t). \end{aligned} \quad (93)$$

Hence the effective Hamiltonian  $\tilde{H}(t)$  takes the form

$$\begin{aligned} \tilde{H}(t) &= i\frac{d}{dt}[U(t) \otimes V(t)]^\dagger (U(t) \otimes V(t)) + (U(t) \otimes V(t))^\dagger H(U(t) \otimes V(t)) = \\ &= -U^\dagger(t)H^\alpha U(t) \otimes V^\dagger(t)V(t) + U^\dagger(t)U(t) \otimes \dot{V}^\dagger(t)V(t) + \\ &\quad + U^\dagger(t)H^\alpha U(t) \otimes V^\dagger(t)V(t) + U^\dagger(t)U(t) \otimes V^\dagger(t)H^{\beta \dots \omega}V(t) = \\ &= \mathbb{I}^\alpha \otimes (\dot{V}^\dagger(t)V(t) + V^\dagger(t)H^{\beta \dots \omega}V(t)) = \\ &= \mathbb{I}^\alpha \otimes \tilde{H}^{\beta \dots \omega}. \end{aligned} \quad (94)$$

And thus, (80) becomes

$$j_{jn;im}(t) = 2\text{Im}[\langle \Psi'(t)|P_j^\alpha(0)P_i^\alpha(0) \otimes P_n(0)\tilde{H}^{\beta \dots \omega}P_m(0)|\Psi'(t)\rangle], \quad (95)$$

which is obviously zero for  $j \neq i$ . And therefore, given the choice (53) for the  $t_{jn;im}(t)$ , we find that  $t_{jn;im}(t)$  is zero whenever  $i \neq j$ , providing that  $p_{im}(t) \neq 0$ . Hence we have shown that the freely evolving system,  $\alpha$ , evolves deterministically, following the evolution of the  $P_i^\alpha(t)$  according to  $U(t)$ . (An alternative proof is given by Bacciagaluppi (1996a, Ch. 7, and 1998a).)

The reader might well be wondering why we did not follow an apparently much easier route to the same conclusion. This easier route would have us sum the current,  $j_{jn;im}(t)$ , over the collective index,  $m$ , to find a ‘marginal’ current for  $\alpha$ . We could then calculate directly the transition probabilities for  $\alpha$ , and it would in fact be trivial to show that they are deterministic when  $\alpha$  evolves freely. But this method is fatally flawed: given Bell’s solution (53), a little thought shows that one *cannot* calculate transition probabilities for a subsystem by summing the compound system’s current over the other indices.

## 5.2 General Subsystems

Hence the calculation of transition probabilities for general subsystems of the universe is a non-trivial problem. We present in this section just a summary of minor results on this question. (For further discussion of these and related points about dynamics in modal interpretations, see Bacciagaluppi (1996a, 1998a, 1998b) and Dickson (1995c, 1998).)

First, the result on deterministic evolution of the previous subsection allows one to derive already some (finite-time) transition probabilities for special cases of atomic systems that do interact with their environment. This can be done by using a technique due to Vermaas (1996). Namely, transition probabilities  $p_{ji}(t, s)$  can be derived for an interacting system when either at  $t$  or at  $s$  its definite properties are in one-to-one correlation with the properties of a freely evolving system (which we must assume to be atomic). Thus, for instance, the transition probabilities for an ideal measurement of an arbitrary observable of one of two entangled atomic systems can be readily calculated. Interestingly enough, in this case the dependence of the measurement result on the possessed property of the measured system has the same form as the Born rule of standard quantum mechanics.

Further, one can show (as in Bacciagaluppi (1996a, Ch. 7.4)) that deterministic evolution holds true also of an atomic system that is interacting with its environment, but whose definite properties *commute* with the interaction Hamiltonian. Thus, one can show that an ideal measurement of an *already possessed* property of an atomic system does not disturb the possessed property of the system. Similarly, if the system is *decohered* by its environment, the definite properties of the system will approximately correspond, except in cases of strong near-degeneracy, to the eigenspaces of the decohering observable (which commutes with the interaction Hamiltonian, see *e.g.*, Zurek (1981)). Hence, an atomic system whose only interaction with its environment is by ways of decoherence, will also (at least approximately) follow a deterministic evolution. In particular, if one idealises a measuring apparatus as having a discrete pointer observable that is decohered (again as in Zurek (1981)) and, further, as being an atomic system, then after the measurement is completed, the pointer readings of the apparatus will not exhibit any stochastic jumps.<sup>13</sup>

The results by Vermaas (1996) and this generalization can be used to furnish good examples of the ‘non-Born-like’ form of the transition probabilities (unlike the special case mentioned above), and of the non-Markovian behavior of interacting subsystems (and at the same time indications of when Markovian approximations might be possible). Further, they illustrate how, in these toy models, the possessed properties of atomic systems play the role of *hidden variables*, determining in part the outcomes of measurements on the system. From this point of view, the atomic modal interpretation equipped with our dynamics has many points of contact with the Bohm theory, not least the non-locality resulting from the dependence of the transition probabilities of one system on the actually possessed properties of other systems (analogously to

the guidance equation for one particle depending on the positions of the other particles). Problems arising in the Bohm theory can be developed and discussed also in the context of the modal interpretation, as frame-dependence of trajectories (Dickson and Clifton 1998) and the justification of the distribution postulate, *i.e.*, the ‘initial’ single-time probabilities (Bacciagaluppi, Barrett and Dickson 1997).

In addition, there are ‘standard’ questions from the theory of stochastic processes to be asked. For example, there are, in fact, distinctions to be made among various versions of the Markov property, and we may ask which, if any, hold, and under what conditions. Also, there is a well-developed theory of semi-group formulations of Markov processes. It would be worth seeing how to formulate the processes we describe here in this way, so that the powerful theory of semigroups could be brought to bear on certain questions.

Finally, the work we have done here might be adaptable to other interpretations. In the first place, it might be adaptable to some modal interpretation for continuous observables, although such an interpretation has yet to be fully worked out (for recent work in this direction, see Clifton (1997)). For this case, there is already in place a well-developed theory of diffusion processes (continuous-time Markov processes with continuous state spaces) upon which a dynamics can draw. In this case, the powerful theory of stochastic differential equations could also be useful, and indeed we may hold out the prospect of genuine (generally stochastic) equations of motion for the modal interpretation. And apart from modal interpretations for continuous observables, our work may well apply to other interpretations. For example, it may show how to define a dynamics for the quantum logic interpretation (on which, see Dickson (1998)), or for the many worlds interpretation. Of course, these remarks are somewhat speculative, but what is clear is that there is more interesting work to be done.

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## NOTES

1. For specific proposals, see Van Fraassen (1979; 1991, Ch. 9), Kochen (1985), Dieks (1988, 1989, 1994), Healey (1989), Bub (1992, 1994), and Vermaas and Dieks (1995).
2. The reason is clear: we would like to be able to represent, in a sensible way, probabilities of conjunctions, disjunctions, and negations of events, and to do so, we need some algebraic operations to represent conjunction, disjunction, and negation. (We mention this point because at least one modal interpretation, that of Healey (1989), apparently does *not* choose a set of possible properties that forms an algebra or partial algebra, at least not under the lattice-theoretic operations.)
3. For a statement of the *problem*, see, for example, Albert and Loewer (1990, 1993) and Elby (1993). Discussions of and solutions to the original problem were given by Bacciagaluppi and Hemmo (1994, 1996), Dickson (1994), Healey (1993a, 1993b), and Ruetsche (1995). Recently, however, serious new problems have emerged—see Bacciagaluppi (1996b), Bacciagaluppi, Donald and Vermaas (1995), and Donald (1997)—which, we believe, show the inadequacy of the non-atomic versions.
4. There are as well other constraints that might be imposed. See Clifton (1995a, 1995b, 1996), Dickson (1995a, 1995c), Bub and Clifton (1996), Bub (1997), and Dieks (1995) for discussions. For very general discussions of constraints on the algebraic structure of the set of definite-valued properties see Bell and Clifton (1995), Dickson (1995b, 1996), and Zimba and Clifton (1997).
5. A consideration of dynamics also turns out to be important for the discussion of state preparation followed by measurement. See Bacciagaluppi and Hemmo (1997). In addition, it has been argued elsewhere (Dickson 1995c, 1995d) that one way (though not the only way) to make sense of the modal interpretation’s denial of the projection postulate is to give the *uncollapsed* state dynamical significance (so as not to make it superfluous). The only way to do so convincingly is to exhibit a reasonable dynamics in which the uncollapsed state plays a crucial role. Finally, an account of dynamics is crucial in the consideration of Lorentz-invariance—see Dickson and Clifton (1997).
6. See, however, the discussion by Bell (1976) in the context of the Everett interpretation.
7. The definition of  $p_{ji}(t, s)$  given in (30) also yields an interpretation of the pathological case

$$\sum_j p_{ji}(t, s) < 1. \quad (96)$$

In fact, if (96) holds, one can say there is a nonzero probability for an *infinite* number of jumps to occur in the finite time interval  $[s, t]$ . Solutions with (96) are called *quasi-processes*, or *dishonest*, or *non-conservative* processes. For example, if  $j$  represents the number of individuals in a population,  $\sum_j p_{ji}(t, s) < 1$  means that there is nonzero probability for a transition from  $i$  individuals to infinitely many individuals in the finite time interval  $[s, t]$ .

On the other hand, Feller shows that if the  $t_{ii}(t)$  are bounded uniformly in  $i$  by a function  $\pi(t)$  such that

$$\int_{T_1}^{T_2} \pi(t)^\alpha dt < \infty \quad (97)$$

for some  $\alpha > 1$ , then

$$\sum_j p_{ji}(t, s) = 1. \quad (98)$$

In particular, if  $I$  is finite, then  $\pi(t) := \max_i |t_{ii}|$  is integrable to any power  $\alpha > 1$  on every open interval  $]s, t[$  with  $T_1 < s < t < T_2$ . Thus, (98) is satisfied on any such interval, and consequently on the whole of  $]T_1, T_2[$ .

8. A clarification of ‘free evolution’: it might be supposed that ‘unitary evolution’ and ‘free evolution’ are equivalent in quantum mechanics. They are not. A system evolves freely between times  $s$  and  $t$  if and only if each member of the family of unitary operators  $U(t, t')$  ( $s \leq t' \leq t$ ) that carries the system’s state from time  $t'$  to time  $t$  is a homogenous function of time, *i.e.*, is a function only of the difference  $t - t'$ , in which case  $U(t, t')$  is generated by some time-independent Hamiltonian.

9. Actually, Bell’s expression for  $t_{ji}$  is not exactly (53), but

$$t_{ji} := \begin{cases} \frac{j_{ji}}{p_i} & \text{for } j_{ji} > 0, \\ & \\ 0 & \text{for } j_{ji} \leq 0. \end{cases} \quad (99)$$

The difference arises precisely when  $j_{ji} = p_i = 0$ , where (53) may be infinite, while (99) is zero. Our choice is continuous at these exceptional points.

10. Here is a direct proof. Suppose that for all  $j$ ,

$$\int_{T_0-\varepsilon}^{T_0} \left| \frac{j_{ji}(t)}{p_i(t)} \right| dt < \infty, \quad (100)$$

where  $T_0$  is a zero of  $p_i(t)$ . Then also

$$\begin{aligned} & \left| \int_{T_0-\varepsilon}^{T_0} \frac{\sum_j j_{ij}(t)}{p_i(t)} dt \right| = \\ & = \left| \int_{T_0-\varepsilon}^{T_0} \frac{\sum_j j_{ji}(t)}{p_i(t)} dt \right| \leq \sum_j \int_{T_0-\varepsilon}^{T_0} \left| \frac{j_{ji}(t)}{p_i(t)} \right| dt < \infty, \end{aligned} \quad (101)$$

by (46), the triangle inequality and (100). However,  $\sum_j j_{ij}(t) = \dot{p}_i(t)$ , by (47), and  $\dot{p}_i(t)/p_i(t)$  has a logarithmic, thus non-integrable, singularity at  $T_0$ :

$$\int_{T_0-\varepsilon}^{T_0} \frac{\dot{p}_i(t)}{p_i(t)} dt =$$

$$= - \int_{T_0-\varepsilon}^{T_0} \frac{d}{dt} \log p_i(t) dt = - \log p_i(t) \Big|_{T_0-\varepsilon}^{T_0} = \infty. \quad (102)$$

Thus, at least for some  $j$ , (100) must fail.

11. The example to follow is based on an example suggested to us by David Albert at the Workshop on Quantum Measurement, University of Minnesota, May 1995.

12. From this point of view, we can see the construction of the current in (86) as follows. Two expressions that, when summed over  $i$ , obviously yield the extra term in  $\dot{p}_j(t)$  are

$$\langle \psi(t) | \dot{P}_j(t) P_i(t) | \psi(t) \rangle \quad (103)$$

and

$$\langle \psi(t) | P_i(t) \dot{P}_j(t) | \psi(t) \rangle. \quad (104)$$

Neither (103) nor (104) are in general real, but they are complex conjugates of one another. Hence by adding them, we get a real number, and (using  $P_i(t)\dot{P}_j(t) = -\dot{P}_i(t)P_j(t)$ ) their sum is anti-symmetric and, indeed, just the term that we added to the Schrödinger current to get (86).

13. The requirements of decoherence (and thus macroscopicity) of the pointer and at the same time of atomic behavior (in the sense of our dynamics) will not necessarily go hand in hand. Further, as hinted in note 3, in more realistic (infinite- or high-dimensional) models, not even the definiteness of pointer readings seems given (see in particular Bacciagaluppi (1996b)). As toy models for the dynamics, however, these examples serve their purpose.